DET KONGELIGE NORSKE VIDENSKABERS SELSKAB
FORHANDLINGER BD XXV, 1952 NR 10

517.61

A Set of Formulas for Numerical Integration

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(Fremlagt i Fellesmøtet 10de mars 1952 av herr S. Selberg)

This paper deels with the problem of finding the numerical value of the integral between certain limits of a function which is known at equidistant points.

The problem is most often attacked by the «trapez formula»:

$$(1)\qquad \int_0^n y dx \backsim T_1 = h\left(\frac{1}{2}y_0 + y_1 + y_2 + \ldots. + y_{n-1} + \frac{1}{2}y_n\right)$$

or Simpson's formula:

$$(2)\qquad \int_0^n y\, dx \backsim S = \frac{2}{3}h\left(\frac{1}{2}y_0 + 2y_1 + y_2 + 2y_3 + y_4 + \ldots + 2y_{n-1} + \frac{1}{2}y_n\right)$$

When Simpson's formula is used n must be an even number. Intermediate values of the integral can only be obtained at every second value of x. The alternating weights of the y's throughout the field of integration are not very convenient for the calculation of long series.

We will attack the problem by a slightly different method which to the authors knowledge is new. We draw a parabola of the third degree through four consecutive points, for instance 0 to 3, and calculate the area of the middle interval 1 to 2. We easily find:

$$(3)\qquad \int_1^2 y\, dx \backsim \frac{h}{24}(-y_0 + 13y_1 + 13y_2 - y_3)$$

By summing up the intervals from 0 to n we get:

$$\int_0^n y\, dx \backsim T_3 = \frac{h}{24}(-y_{-1} + 12y_0 + 25y_1 + 24y_2 + 24y_3 + \ldots$$
$$+ 24y_{n-2} + 25y_{n-1} + 12y_n - y_{n+1})$$

$$(4)\qquad = T_1 + \frac{h}{24}(-y_{-1} + y_1 + y_{n-1} - y_{n+1})$$

We introduce the following notations:

$$\overline{y}_i = \frac{1}{2}(y_{i-\frac{1}{2}} + y_{i+\frac{1}{2}}),\ \Delta y_i = y_{i+\frac{1}{2}} - y_{i-\frac{1}{2}},\ \Delta^2 y_i = \Delta y_{i+\frac{1}{2}} - \Delta y_{i-\frac{1}{2}}$$

and so on.

We then get for the interval i to $i+1$:

$$(5)\qquad \int_i^{i+1} y\, dx \backsim \Delta T_{3,i+\frac{1}{2}} = h\left(\overline{y}_{i+\frac{1}{2}} - \frac{1}{12}\Delta^2 \overline{y}_{i+\frac{1}{2}}\right)$$

and for the total integral

$$(6)\qquad \int_0^n y\, dx \backsim T_3 = T_1 + \frac{h}{12}(\Delta\overline{y}_0 - \Delta\overline{y}_n)$$

The values of y appear in the formula with equal weights except near the limits of the field of integration, and intermediate values of the integral can easily be obtained at every value of x.

It is not difficult to extend this method to parabolas of higher degrees. We get for the fifth and seventh degree respectively:

$$(7)\qquad \Delta T_{5,i+\frac{1}{2}} = \Delta T_{3,i+\frac{1}{2}} + \frac{11\,h}{720}\Delta^4\overline{y}_{i+\frac{1}{2}}$$

$$(8)\qquad \Delta T_{7,i+\frac{1}{2}} = \Delta T_{5,i+\frac{1}{2}} - \frac{191\,h}{60480}\Delta^6\overline{y}_{i+\frac{1}{2}}$$

$$(9)\qquad T_5 = T_3 - \frac{11h}{720}(\Delta^3\overline{y}_0 - \Delta^3\overline{y}_n)$$

$$(10)\qquad T_7 = T_5 + \frac{191\,h}{60480}(\Delta^5\overline{y}_0 - \Delta^5\overline{y}_n)$$

As an example let us test the result

$$\int_0^6 \frac{12}{\pi}\cos\frac{12}{\pi}x\, dx = 1,\ h = 1$$

We use the values of $\cos x = 0°,\ 15°, \ldots\ 90°$ and get:

$T_1 = 0{,}9943\ ,\quad T_3 = 0{,}999928\ ,\quad T_5 = 0{,}99999899\ ,\quad T_7 = 0{,}999999985\ .$

Simpson's formula gives:

$S = 1{,}000026$, a better approximation than T_3, but far behind T_5.

Expressions for the rest terms may be obtained follows: If the fourth derivative of y, $y^{(4)}$, is constant in the field of integration we easily get:

$R_3 = \int_0^n y\,dx - T_3 = \frac{11}{720} L h^3 y^{(4)}$, where $L = nh$, the total interval of x. If y is not a constant we have:

$R_3 = \frac{11}{720} L h^3 y^{(4)}_m$, where $y^{(4)}_m$ is the value of $y^{(4)}$ at some value $x = m$, within the boundaries. We get correspondingly:

(11) $$R_1 = -\frac{1}{12} L\, h^2 y''_m$$

(12) $$R_3 = \frac{11}{720} L\, h^4 y^{(4)}_m$$

(13) $$R_5 = -\frac{191}{60480} L\, h^6 y^{(6)}_m$$

(14) $$R_7 = \frac{2497}{3628800} L\, h^8 y^{(8)}_m\ .$$

The corresponding expression for Simpson's formula is:

(15) $$R_S = -\frac{1}{180} L\, h^4 y^{(4)}_m\ .$$

The expression for T_3, T_5 and T_7 contain two or more values of y outside the field of integration. If these are not known, they may be extrapolated by assuming the constancy of $\overline{y}'''$ when T_3 is used, of $\overline{y}^{(5)}$ when T_5 is used and so on. We then arrive at Gregory's formulas for the total integral.

The result of the integration can be made more accurate either by diminishing h, or by taking differences of higher order into account. When the series of differences tend to show irregularities there is generally no more to be gained in this way.

In some cases the values of y are known in the middle of the intervals over which the integrals are wanted. We may then proceed in the same way as above, and get with corresponding notations:

(16) $$\Delta T_{0,i+\frac{1}{2}} = h\, y_{i+\frac{1}{2}}$$

(17) $$\Delta T_{2,i+\frac{1}{2}} = \Delta T_{0,i+\frac{1}{2}} + \frac{h}{24} \Delta^2 y_{i+\frac{1}{2}}$$

(18) $$\Delta T_{4,i+\frac{1}{2}} = \Delta T_{2,i+\frac{1}{2}} - \frac{17\,h}{5760} \Delta^4 y_{i+\frac{1}{2}}$$

(19) $$\Delta T_{6,i+\frac{1}{2}} = \Delta T_{4,i+\frac{1}{2}} + \frac{367\,h}{967680} \Delta^6 y_{i+\frac{1}{2}}\ ,\quad \text{and}$$

(20) $$\int_0^n y\,dx \backsim T_0 = h \sum_0^{n-1} y_{i+\frac{1}{2}}$$

(21) $$T_2 = T_0 - \frac{h}{24} (\Delta y_0 - \Delta y_n)$$

(22) $$T_4 = T_2 + \frac{17\,h}{5760} (\Delta^3 y_0 - \Delta^3 y_n)$$

(23) $$T_6 = T_4 - \frac{367\,h}{967680} (\Delta^5 y_0 - \Delta^5 y_n)$$

with the rest terms:

(24) $$R_0 = \frac{1}{24} L h^2 y''_m$$

(25) $$R_2 = -\frac{17}{5760} L\, h^4 y^{(4)}_m$$

(26) $$R_4 = \frac{367}{967680} L\, h^6 y^{(6)}_m$$

(27) $$R_6 = -\frac{27859}{464466400} L\, h^8 y^{(8}\ .$$

The same test as above, using the values of $\cos x$ for $7\frac{1}{2}°$, $22\frac{1}{2}°$, $37\frac{1}{2}°, \ldots\ 82\frac{1}{2}°$, gives:

$T_0 = 1{,}0053\ ,\quad T_2 = 1{,}000014\ ,\quad T_4 = 1{,}00000013\ ,\quad T_6 = 1{,}000000013\ .$

Trykt 5te mai 1952
I kommisjon hos F. Bruns Bokhandel
Aktietrykkeriet i Trondhjem